

APPROXIMATION OF CLASSES OF FUNCTIONS DEFINED BY
A GENERALIZED r -TH MODULUS OF SMOOTHNESS

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Dedicated to Professor P. L. Ul'yanov on the occasion of his 70-th birthday

ABSTRACT. In this paper, a k -th generalized modulus of smoothness is defined based on an asymmetric operator of generalized translation and a theorem is proved about the coincidence of class of functions defined by this modulus and a class of functions having given order of best approximation by algebraic polynomials.

Introduction. In paper [4], an asymmetric operator of generalized translation was introduced and by means of it the corresponding generalized modulus of smoothness of first order was defined. Then a theorem was proved about coincidence of the class of functions defined by this modulus with the class of functions having a given order of best approximation by algebraic polynomials.

In the present paper, analogous results are obtained for the generalized modulus of smoothness of order r . In addition, the space in which the theorem of coincidence of the corresponding classes of functions holds true is widened.

1. For $1 \leq p < \infty$, as usual, L_p denotes the set all measurable functions f on $[-1, 1]$ for which

$$\|f\|_p = \left(\int_{-1}^1 |f(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, L_∞ is the space of all continuous functions f on $[-1, 1]$ with a norm

$$\|f\|_\infty = \max_{-1 \leq x \leq 1} |f(x)|.$$

Denote by $L_{p,\alpha,\beta}$ the set of functions f such that $f(x)(1-x)^\alpha(1+x)^\beta \in L_p$, and set

$$\|f\|_{p,\alpha,\beta} = \|f(x)(1-x)^\alpha(1+x)^\beta\|_p.$$

By $E_n(f)_{p,\alpha,\beta}$ we denote the best approximation of $f \in L_{p,\alpha,\beta}$ by algebraic polynomials of degree not greater than $n-1$ in $L_{p,\alpha,\beta}$ metrics, that is,

$$E_n(f)_{p,\alpha,\beta} = \inf_{P_n \in \mathcal{P}_n} \|f - P_n\|_{p,\alpha,\beta},$$

where \mathcal{P}_n is the set of algebraic polynomials of degree not greater than $n-1$.

By $E(p, \alpha, \beta, \lambda)$ we denote the class of functions $f \in L_{p,\alpha,\beta}$ satisfying the condition

$$E_n(f)_{p,\alpha,\beta} \leq Cn^{-\lambda},$$

where $\lambda > 0$ and C is a constant not depending on n ($n \in \mathbb{N}$).

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For functions f we define the operator of *generalized translation* $\hat{T}_t(f, x)$ by

$$\hat{T}_t(f, x) = \frac{1}{\pi(1-x^2)} \int_0^\pi \left(1 - \left(x \cos t - \sqrt{1-x^2} \sin t \cos \varphi \right)^2 - 2 \sin^2 t \sin^2 \varphi \right. \\ \left. + 4(1-x^2) \sin^2 t \sin^4 \varphi \right) f(x \cos t - \sqrt{1-x^2} \sin t \cos \varphi) d\varphi.$$

By means of this operator of generalized translation we define the *generalized difference* of order r by

$$\Delta_t^1(f, x) = \Delta_t(f, x) = \hat{T}_t(f, x) - f(x),$$

$$\Delta_{t_1, \dots, t_r}^r(f, x) = \Delta_{t_r} \left(\Delta_{t_1, \dots, t_{r-1}}^{r-1}(f, x), x \right) \quad (r = 2, 3, \dots),$$

and the *generalized modulus of smoothness* of order r by

$$\hat{\omega}_r(f, \delta)_{p, \alpha, \beta} = \sup_{|t_i|_{i=1, \dots, r} \leq \delta} \left\| \Delta_{t_1, \dots, t_r}^r(f, x) \right\|_{p, \alpha, \beta} \quad (r = 1, 2, \dots).$$

Consider the class $H(p, \alpha, \beta, r, \lambda)$ of functions $f \in L_{p, \alpha, \beta}$ satisfying the condition

$$\hat{\omega}_r(f, \delta)_{p, \alpha, \beta} \leq C \delta^\lambda,$$

where $\lambda > 0$ and C is a constant not depending on δ .

The aim of the present paper is to prove the following statement

Theorem 1.1. *Let p, α, β and r be given numbers such that $1 \leq p \leq \infty, r \in \mathbb{N}$;*

$$\begin{aligned} & \frac{1}{2} < \alpha \leq 2, & \frac{1}{2} < \beta \leq 2 & \quad \text{for } p = 1, \\ & 1 - \frac{1}{2p} < \alpha < 3 - \frac{1}{p}, & 1 - \frac{1}{2p} < \beta < 3 - \frac{1}{p} & \quad \text{for } 1 < p < \infty, \\ & 1 \leq \alpha < 3, & 1 \leq \beta < 3 & \quad \text{for } p = \infty. \end{aligned}$$

Then, for any λ satisfying the condition

$$\lambda_0 = 2 \max \left(|\alpha - \beta|, \alpha - \frac{3}{2} + \frac{1}{2p}, \beta - \frac{3}{2} + \frac{1}{2p} \right) < \lambda < 2r$$

the class $H(p, \alpha, \beta, r, \lambda)$ coincides with the class $E(p, \alpha, \beta, \lambda)$.

The validity of Theorem 1.1 will follow from the validity of Theorems 4.3 and 4.4, which we are going to prove below.

2. Put $y = \cos t, z = -\cos \varphi$ in the definition of $\hat{T}_t(f, x)$ and denote the resulting operator by $T_y(f, x)$. Let us rewrite it in the form

$$T_y(f, x) = \frac{1}{\pi(1-x^2)} \int_{-1}^1 \left(1 - R^2 - 2(1-y^2)(1-z^2) \right. \\ \left. + 4(1-x^2)(1-y^2)(1-z^2)^2 \right) f(R) \frac{dz}{\sqrt{1-z^2}},$$

where $R = xy - z\sqrt{1-x^2}\sqrt{1-y^2}$. We define the operator of generalized translation of order r by

$$T_y^1(f, x) = T_y(f, x),$$

$$T_{y_1, \dots, y_r}^r(f, x) = T_{y_r} \left(T_{y_1, \dots, y_{r-1}}^{r-1}(f, x), x \right) \quad (r = 2, 3, \dots).$$

By $P_\nu^{(\alpha, \beta)}(x)$ ($\nu = 0, 1, \dots$) we denote the Jacobi polynomials, i.e. algebraic polynomials of degree ν orthogonal on the segment $[-1, 1]$ with a weight $(1-x)^\alpha(1+x)^\beta$ and normalized by the condition $P_\nu^{(\alpha, \beta)}(1) = 1$ ($\nu = 0, 1, \dots$).

For any integrable function f on $[-1, 1]$ with a weight $(1 - x^2)^2$, we denote by $a_n(f)$ the Fourier–Jacobi coefficients of f with respect to the system of Jacobi polynomials $\{P_n^{(2,2)}(x)\}_{n=0}^\infty$, i.e.

$$a_n(f) = \int_{-1}^1 f(x) P_n^{(2,2)}(x) (1 - x^2)^2 dx \quad (n = 0, 1, \dots).$$

Introduce certain operators which will play an auxiliary role later on. First we set

$$T_{1;y}(f, x) = \frac{1}{\pi(1 - x^2)} \int_{-1}^1 (1 - R^2 - 2(1 - y^2)(1 - z^2)) f(R) \frac{dz}{\sqrt{1 - z^2}},$$

$$T_{2;y}(f, x) = \frac{8}{3\pi} \int_{-1}^1 (1 - z^2)^2 f(R) \frac{dz}{\sqrt{1 - z^2}},$$

where $R = xy - z\sqrt{1 - x^2}\sqrt{1 - y^2}$, and then define the corresponding operators of order r by

$$T_{k;y}^1(f, x) = T_{k;y}(f, x),$$

$$T_{k;y_1, \dots, y_r}^r(f, x) = T_{k;y_r} \left(T_{k;y_1, \dots, y_{r-1}}^{r-1}(f, x), x \right) \quad (r = 2, 3, \dots)$$

for $k = 1, 2$.

3.

Lemma 3.1. *Let $f \in L_{p,\alpha,\beta}$ and let the numbers $p, \alpha, \beta, \rho, \sigma$ and λ be such that $1 \leq p \leq \infty, \rho \geq 0, \sigma \geq 0, \lambda > \lambda_0 = 2 \max\{\rho, \sigma\}$;*

$$\alpha > -\frac{1}{p}, \quad \beta > -\frac{1}{p} \quad \text{for } 1 \leq p < \infty,$$

$$\alpha \geq 0, \quad \beta \geq 0 \quad \text{for } p = \infty.$$

If there exists a sequence of algebraic polynomials $\{P_{2^n}(x)\}_{n=0}^\infty$ such that

$$\|f - P_{2^n}\|_{p,\alpha+\rho,\beta+\sigma} \leq \frac{C_1}{2^{n\lambda}},$$

then the following inequalities also hold true

$$\|f - P_{2^n}\|_{p,\alpha,\beta} \leq \frac{C_2}{2^{n(\lambda-\lambda_0)}} \quad (n = 1, 2, \dots),$$

where the constants C_1 and C_2 do not depend on n .

Lemma 3.1 was proved in [2].

Lemma 3.2. *Let $P_n(x)$ be an algebraic polynomial of degree not greater than $n - 1$, $1 \leq p \leq \infty, \rho \geq 0, \sigma \geq 0$. Assume that*

$$\alpha > -\frac{1}{p}, \quad \beta > -\frac{1}{p} \quad \text{for } 1 \leq p < \infty,$$

$$\alpha \geq 0, \quad \beta \geq 0 \quad \text{for } p = \infty.$$

Then

$$\|P'_n(x)\|_{p,\alpha+\frac{1}{2},\beta+\frac{1}{2}} \leq C_1 n \|P_n\|_{p,\alpha,\beta},$$

$$\|P_n\|_{p,\alpha,\beta} \leq C_2 n^{2 \max(\rho,\sigma)} \|P_n\|_{p,\alpha+\rho,\beta+\sigma},$$

where the constants C_1 and C_2 do not depend on n .

Lemma was proved in [1].

Lemma 3.3. *The operators $T_{1;y}$ and $T_{2;y}$ have the following properties*

$$\begin{aligned} T_{1;y} \left(P_\nu^{(2,2)}, x \right) &= P_\nu^{(2,2)}(x) P_{\nu+2}^{(0,0)}(y), \\ T_{2;y} \left(P_\nu^{(2,2)}, x \right) &= P_\nu^{(2,2)}(x) P_\nu^{(2,2)}(y) \end{aligned}$$

for $\nu = 0, 1, \dots$

Lemma 3.3 was proved in [4].

Lemma 3.4. *Let $g(x)T_{k;y}(f, x) \in L_{1,2,2}$ for every y . Then for $k = 1, 2$ the following equality holds true*

$$\int_{-1}^1 f(x)T_{k;y}(g, x) (1-x^2)^2 dx = \int_{-1}^1 g(x)T_{k;y}(f, x) (1-x^2)^2 dx.$$

Proof. Let $k = 1$ and

$$\begin{aligned} I_1 &:= \int_{-1}^1 f(x)T_{1;y}(g, x) (1-x^2)^2 dx \\ &= \frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 f(x)g(R) (1-R^2 - 2(1-y^2)(1-z^2)) (1-x^2) \frac{dz dx}{\sqrt{1-z^2}}, \end{aligned}$$

where $R = xy - z\sqrt{1-x^2}\sqrt{1-y^2}$. Performing change of variables in the double integral by the formulas

$$\begin{aligned} x &= Ry + V\sqrt{1-R^2}\sqrt{1-y^2}, \\ z &= -\frac{R\sqrt{1-y^2} - Vy\sqrt{1-R^2}}{\sqrt{1 - \left(Ry + V\sqrt{1-R^2}\sqrt{1-y^2}\right)^2}}, \end{aligned} \quad (3.1)$$

we get

$$\begin{aligned} I_1 &= \frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 (1-R^2) f \left(Ry + V\sqrt{1-R^2}\sqrt{1-y^2} \right) g(R) \\ &\quad \times \left(1 - \left(Ry + V\sqrt{1-R^2}\sqrt{1-y^2} \right)^2 - 2(1-y^2)(1-V^2) \right) \frac{dV dR}{\sqrt{1-V^2}} \\ &= \int_{-1}^1 g(R)T_{1;y}(f, R) (1-R^2)^2 dR, \end{aligned}$$

which proves the equality of the lemma for $k = 1$.

Let $k = 2$ and

$$\begin{aligned} I_2 &:= \int_{-1}^1 f(x)T_{2;y}(g, x) (1-x^2)^2 dx \\ &= \frac{8}{3\pi} \int_{-1}^1 \int_{-1}^1 f(x)g(R) (1-x^2)^2 (1-z^2)^2 \frac{dz dx}{\sqrt{1-z^2}}. \end{aligned}$$

Performing again the change (3.1) in the double integral we get

$$\begin{aligned} I_2 &= \frac{8}{3\pi} \int_{-1}^1 \int_{-1}^1 f \left(Ry + V\sqrt{1-R^2}\sqrt{1-y^2} \right) g(R) (1-R^2)^2 \\ &\quad \times (1-V^2)^2 \frac{dV dR}{\sqrt{1-V^2}} = \int_{-1}^1 g(R)T_{2;y}(f, R) (1-R^2)^2 dR. \end{aligned}$$

Lemma 3.4 is proved. \square

Corollary 3.1. *If $f \in L_{1,2,2}$, then for every $r \in \mathbb{N}$ we have $T_{k;r_1,\dots,r_y}^r(f, x) \in L_{1,2,2}$ ($k = 1, 2$).*

Proof. Put $g(x) \equiv 1$ on $[-1, 1]$. Taking into account that by Lemma 3.3

$$\begin{aligned} T_{1;y}(1, x) &= T_{1;y}\left(P_0^{(2,2)}, x\right) = P_0^{(2,2)}(x)P_2^{(0,0)}(y) = \frac{3}{2}y^2 - \frac{1}{2}, \\ T_{2;y}(1, x) &= 1, \end{aligned}$$

we clearly have $f(x)T_{k;y}(1, x) \in L_{1,2,2}$ ($k = 1, 2$). Hence, applying Lemma 3.4 we derive the relation

$$\int_{-1}^1 T_{k;y}(f, x) (1 - x^2)^2 dx = \int_{-1}^1 f(x)T_{k;y}(1, x) (1 - x^2)^2 dx \quad (k = 1, 2),$$

which implies that $T_{k;y}(f, x) \in L_{1,2,2}$. Now the corollary can be proved by induction. \square

Lemma 3.5. *Let f be an integrable function on $[-1, 1]$ with a weight $(1 - x^2)^2$. For every natural number n the following equality holds true*

$$\int_{-1}^1 T_{1;y}(f, x) P_n^{(1,1)}(y) dy = \sum_{m=0}^{n-2} a_m(f) \gamma_m(x),$$

where $\gamma_m(x)$ is an algebraic polynomial of degree not greater than $n-2$, and $\gamma_m(x) \equiv 0$ for $n = 0$ or $n = 1$.

Lemma 3.5 was proved in [4].

Lemma 3.6. *Let q and m be given natural numbers. Let f be an integrable function on $[-1, 1]$ with a weight $(1 - x^2)^2$. Then for every natural numbers l and r ($l \leq r$) the function*

$$Q_1^{(l)}(x) = \int_0^\pi \dots \int_0^\pi T_{1;\cos t_1, \dots, \cos t_l}^l(f, x) \prod_{s=1}^r \left(\frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r$$

is an algebraic polynomial of degree not greater than $(q+2)(m-1)$.

Proof. In this paper we denote, for simplicity,

$$A(t) := \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4}.$$

Since

$$A(t_s) = \sum_{k=0}^{(q+2)(m-1)} a_k \cos kt_s = \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k,$$

it follows that

$$\begin{aligned} A(t_s) \sin^2 t_s &= \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k (1 - \cos^2 t_s) = \sum_{k=0}^{(q+2)(m-1)+2} c_k (\cos t_s)^k \\ &= \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k P_k^{(1,1)}(\cos t_s) \quad (s = 1, 2, \dots, r). \end{aligned}$$

Hence we have

$$Q_1^{(l)}(x) = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \int_0^\pi \dots \int_0^\pi \prod_{\substack{s=1 \\ s \neq l}}^r A(t_s) \sin^3 t_s dt_1 \dots dt_{l-1} dt_{l+1} \dots dt_r \\ \times \int_0^\pi T_{1;\cos t_1, \dots, \cos t_l}^l(f, x) P_k^{(1,1)}(\cos t_l) \sin t_l dt_l.$$

Let

$$\varphi_{l,k}(x) := \int_0^\pi T_{1;\cos t_1, \dots, \cos t_l}^l(f, x) P_k^{(1,1)}(\cos t_l) \sin t_l dt_l \\ = \int_0^\pi T_{1;\cos t_l} \left(T_{1;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, x), x \right) P_k^{(1,1)}(\cos t_l) \sin t_l dt_l.$$

Substituting $y = \cos t_l$ we obtain

$$\varphi_{l,k}(x) = \int_{-1}^1 T_{1;y} \left(T_{1;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, x), x \right) P_k^{(1,1)}(y) dy.$$

Then, by Lemma 3.5,

$$\varphi_{l,k}(x) = \sum_{m=0}^{k-2} \gamma_m(x) \int_{-1}^1 T_{1;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, R) P_m^{(2,2)}(R) (1-R^2)^2 dR.$$

On the bases of Corollary 3.1, we conclude that $T_{1;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, R) \in L_{1,2,2}$. Applying now $l-1$ times Lemma 3.3 we obtain

$$\varphi_{l,k}(x) \\ = \sum_{m=0}^{k-2} \gamma_m(x) \int_{-1}^1 T_{1;\cos t_1, \dots, \cos t_{l-2}}^{l-2}(f, R) T_{1;\cos t_{l-1}} \left(P_m^{(2,2)}, R \right) (1-R^2)^2 dR \\ = \sum_{m=0}^{k-2} \gamma_m(x) P_{m+2}^{(0,0)}(\cos t_{l-1}) \int_{-1}^1 T_{1;\cos t_1, \dots, \cos t_{l-2}}^{l-2}(f, R) P_m^{(2,2)}(R) (1-R^2)^2 dR \\ = \sum_{m=0}^{k-2} \gamma_m(x) P_{m+2}^{(0,0)}(\cos t_1) \dots P_{m+2}^{(0,0)}(\cos t_{l-1}) \int_{-1}^1 f(R) P_m^{(2,2)}(R) (1-R^2)^2 dR \\ = \sum_{m=0}^{k-2} \gamma_m(x) a_m(f) \prod_{s=1}^{l-1} P_{m+2}^{(0,0)}(\cos t_s),$$

where $a_m(f)$ is the Fourier–Jacobi coefficient of the function f with respect to the system $\left\{ P_m^{(2,2)}(x) \right\}_{m=0}^\infty$. Substituting the last expression of $\varphi_{l,k}(x)$ in the formula above for $Q_1^{(l)}(x)$ we get

$$Q_1^{(l)}(x) = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \sum_{m=0}^{k-2} \beta_m \gamma_m(x).$$

Since $\gamma_m(x)$ is from \mathcal{P}_{k-1} for $k \geq 2$ and $\gamma_m(x) \equiv 0$ for $k = 0$ and $k = 1$, then the last equality yields that $Q_1^{(l)}(x)$ is an algebraic polynomial of degree not greater than $(q+2)(m-1)$.

Lemma 3.6 is proved. \square

Lemma 3.7. *Let q and m be given natural numbers. Let f be an integrable function on $[-1, 1]$ with a weight $(1 - x^2)^2$. For every natural numbers l and r ($l \leq r$) the function*

$$Q_2^{(l)}(x) = \int_0^\pi \dots \int_0^\pi T_{2;\cos t_1, \dots, \cos t_l}^l(f, x) \prod_{s=1}^r A(t_s) \sin^5 t_s dt_1 \dots dt_r$$

is an algebraic polynomial of degree not greater than $(q+2)(m-1)$.

Proof. As shown in Lemma 3.6,

$$A(t_s) = \sum_{k=0}^{(q+2)(m-1)} b_k(\cos t_s)^k = \sum_{k=0}^{(q+2)(m-1)} \beta_k P_k^{(2,2)}(\cos t_s) \quad (s = 1, 2, \dots, r).$$

Hence

$$\begin{aligned} Q_2^{(l)}(x) &= \sum_{k=0}^{(q+2)(m-1)} \beta_k \int_0^\pi \dots \int_0^\pi \prod_{\substack{s=1 \\ s \neq l}}^r A(t_s) \sin^5 t_s dt_1 \dots dt_{l-1} dt_{l+1} \dots dt_r \\ &\quad \times \int_0^\pi T_{2;\cos t_1, \dots, \cos t_l}^l(f, x) P_k^{(2,2)}(\cos t_l) \sin^5 t_l dt_l. \end{aligned}$$

Let

$$\begin{aligned} \psi_{l,k}(x) &:= \int_0^\pi T_{2;\cos t_1, \dots, \cos t_l}^l(f, x) P_k^{(2,2)}(\cos t_l) \sin^5 t_l dt_l \\ &= \int_0^\pi T_{2;\cos t_l} \left(T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, x), x \right) P_k^{(2,2)}(\cos t_l) \sin^5 t_l dt_l. \end{aligned}$$

Substituting $y = \cos t_l$ we obtain

$$\psi_{l,k}(x) = \int_{-1}^1 T_{2;y} \left(T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, x), x \right) P_k^{(2,2)}(y) (1 - y^2)^2 dy.$$

Since the operator $T_{2;y}(f, x)$ is symmetric with respect to x and y (i.e. $T_{2;y}(g, x) = T_{2;x}(g, y)$ for every function g), we have

$$\psi_{l,k}(x) = \int_{-1}^1 T_{2;x} \left(T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, y), y \right) P_k^{(2,2)}(y) (1 - y^2)^2 dy.$$

Note that, in view of Corollary 3.1, $T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, y) \in L_{1,2,2}$. Then, by Lemma 3.4,

$$\psi_{l,k}(x) = \int_{-1}^1 T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, y) T_{2;x} \left(P_k^{(2,2)}, y \right) (1 - y^2)^2 dy.$$

Using the property of the operator $T_{2;x}$ described in Lemma 3.3 we get

$$\psi_{l,k}(x) = P_k^{(2,2)}(x) \int_{-1}^1 T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, y) P_k^{(2,2)}(y) (1 - y^2)^2 dy.$$

Now we apply $l-1$ times Lemma 3.3 and arrive at the expression

$$\begin{aligned} \psi_{l,k}(x) &= P_k^{(2,2)}(x) P_k^{(2,2)}(\cos t_1) \dots P_k^{(2,2)}(\cos t_{l-1}) \\ &\quad \times \int_{-1}^1 f(y) P_k^{(2,2)}(y) (1 - y^2)^2 dy = P_k^{(2,2)}(x) a_k(f) \prod_{s=1}^{l-1} P_k^{(2,2)}(\cos t_s). \end{aligned}$$

where $a_k(f)$ is the Fourier–Jacobi coefficient of the function f with respect to the system $\left\{P_k^{(2,2)}(x)\right\}_{k=0}^{\infty}$. Substituting the last expression of $\psi_{l,k}(x)$ into the formula for $Q_2^{(l)}(x)$ above, we finally get

$$Q_2^{(l)}(x) = \sum_{k=0}^{(q+2)(m-1)} \delta_k P_k^{(2,2)}(x).$$

Since $P_k^{(2,2)}(x)$ belongs to \mathcal{P}_{k+1} , it is seen from the last identity that $Q_2^{(l)}(x)$ is an algebraic polynomial of degree not greater than $(q+2)(m-1)$.

The lemma is proved. \square

Lemma 3.8. *The operator T_y has the following properties*

- (1) $T_y(f, x)$ is linear on f ;
- (2) $T_1(f, x) = f(x)$;
- (3) $T_y(P_n^{(2,2)}, x) = P_n^{(2,2)}(x)R_n(y) \quad (n = 0, 1, \dots)$,
where $R_n(y) = P_{n+2}^{(0,0)}(y) + \frac{3}{2}(1-y^2)P_n^{(2,2)}(y)$;
- (4) $T_y(1, x) = 1$;
- (5) $a_k(T_y(f, x)) = R_k(y)a_k(f) \quad (k = 0, 1, \dots)$.

Lemma 3.8 was proved in [4].

Corollary 3.2. *If $P_n(x)$ is an algebraic polynomial of degree not greater than $n-1$, then for every natural number r and any fixed y_1, y_2, \dots, y_r , the functions $T_{y_1, \dots, y_r}^r(P_n, x)$ are algebraic polynomials of x of degree not greater than $n-1$.*

Lemma 3.9. *If $-1 \leq x \leq 1$, $-1 \leq z \leq 1$, $0 \leq t \leq \pi$ and $R = x \cos t - z\sqrt{1-x^2} \times \sin t$, then $-1 \leq R \leq 1$ and*

$$\begin{aligned} (1-x^2)(1-z^2) &\leq (1-R^2), \\ (1-y^2)(1-z^2) &\leq (1-R^2), \\ \left(x\sqrt{1-y^2} + yz\sqrt{1-x^2}\right)^2 &\leq (1-R^2), \\ 1-x^2 &\leq C(1-R^2+t^2), \\ 1-x &\leq C(1-R+t^2), \\ 1+x &\leq C(1+R+t^2), \end{aligned}$$

where $y = \cos t$ and C is an absolute constant.

Lemma 3.9 was proved in [4] and [3].

Lemma 3.10. *Let p, α, β and γ be given numbers such that $1 \leq p \leq \infty$, $\gamma = \min\{\alpha, \beta\}$, and*

$$\begin{aligned} \gamma &> 1 - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty, \\ \gamma &\geq 1 \quad \text{for } p = \infty. \end{aligned}$$

Let ε , $0 < \varepsilon < \frac{1}{2}$, be an arbitrary number. Define

$$\gamma_1 = \begin{cases} \alpha - \beta, & \text{if } \alpha > \beta \\ 0, & \text{if } \alpha \leq \beta, \end{cases} \quad \gamma_2 = \begin{cases} 0, & \text{if } \alpha > \beta \\ \beta - \alpha, & \text{if } \alpha \leq \beta, \end{cases}$$

and for $1 < p \leq \infty$ let

$$\gamma_3 = \begin{cases} \gamma - \frac{3}{2} + \frac{1}{2p} + \varepsilon, & \text{if } \gamma \geq \frac{3}{2} - \frac{1}{2p} \\ 0, & \text{if } \gamma < \frac{3}{2} - \frac{1}{2p}, \end{cases}$$

while, for $p = 1$,

$$\gamma_3 = \begin{cases} \gamma - 1, & \text{if } \gamma \geq 1 \\ 0, & \text{if } \gamma < 1. \end{cases}$$

Let $R = x \cos t - z \sqrt{1 - x^2} \sin t$. Then, for every measurable function f on $[-1, 1]$ the following inequality holds

$$\begin{aligned} & \left\| \frac{1}{1 - x^2} \int_{-1}^1 (1 - R^2) |f(R)| \frac{dz}{\sqrt{1 - z^2}} \right\|_{p, \alpha, \beta} \\ & \leq C \left(\|f\|_{p, \alpha, \beta} + t^{2(\gamma_1 + \gamma_2)} \|f\|_{p, \alpha - \gamma_1, \beta - \gamma_2} + t^{2\gamma_3} \|f\|_{p, \alpha - \gamma_3, \beta - \gamma_3} \right. \\ & \quad \left. + t^{2(\gamma_1 + \gamma_2 + \gamma_3)} \|f\|_{p, \alpha - \gamma_1 - \gamma_3, \beta - \gamma_2 - \gamma_3} \right), \end{aligned}$$

where the constant C does not depend on f and t .

Proof. If at least one of the terms on the right-hand side of the inequality is not finite, then the lemma is obvious.

Suppose now that all the terms on the right-hand side of the inequality are finite.

Let $\alpha \geq \beta$. We first consider the case $1 \leq p < \infty$. Clearly

$$\begin{aligned} (3.2) \quad I &:= \left\| \frac{1}{1 - x^2} \int_{-1}^1 (1 - R^2) |f(R)| \frac{dz}{\sqrt{1 - z^2}} \right\|_{p, \alpha, \beta}^p \\ &= \int_{-1}^1 \left| \int_{-1}^1 |f(R)| (1 - z^2)^{-1/2} (1 - R^2) dz \right|^p (1 - x)^{p(\alpha-1)} (1 + x)^{p(\beta-1)} dx. \end{aligned}$$

If $p = 1$, then

$$I \leq \int_{-1}^1 \int_{-1}^1 |f(R)| \delta dz dx,$$

where

$$\delta = (1 - z^2)^{-1/2} (1 - R^2) (1 - x^2)^{\beta-1} (1 - x)^{\alpha-\beta}.$$

Let $\beta < 1$. Then, in view of Lemma 3.9,

$$\begin{aligned} \delta &= (1 - z^2)^{\frac{1}{2}-\beta} ((1 - z^2) (1 - x^2))^{\beta-1} (1 - R^2) (1 - x)^{\alpha-\beta} \\ &\leq C_1 (1 - z^2)^{-1/2} ((1 - z^2) (1 - x^2))^{\beta-1} (1 - R^2) (1 - R + t^2)^{\alpha-\beta} \\ &= C_1 \delta_1(x, z, R). \end{aligned}$$

Suppose that $\beta \geq 1$. Making use of Lemma 3.9 we see that

$$\begin{aligned} \delta &\leq C_2 (1 - z^2)^{-1/2} (1 - R^2) (1 - R^2 + t^2)^{\beta-1} (1 - R + t^2)^{\alpha-\beta} \\ &= C_2 \delta_2(x, z, R). \end{aligned}$$

Incorporating these estimates for δ we get the inequality

$$I \leq C_3 \int_{-1}^1 \int_{-1}^1 |f(R)| \delta_k(x, z, R) dz dx \quad (k = 1, 2),$$

which, after the change of variables (3.1), takes the form

$$I \leq C_3 \int_{-1}^1 \int_{-1}^1 |f(R)| \delta_k(R, V, R) dV dR \quad (k = 1, 2).$$

Set

$$\begin{aligned} \hat{\delta}_1(R) &:= (1 - R^2)^\beta (1 - R + t^2)^{\alpha-\beta} \\ &\leq C_4 \left((1 - R)^\alpha (1 + R)^\beta + t^{2(\alpha-\beta)} (1 - R^2)^\beta \right), \end{aligned}$$

$$\begin{aligned}\hat{\delta}_2(R) &:= (1 - R^2) (1 - R^2 + t^2)^{\beta-1} (1 - R + t^2)^{\alpha-\beta} \\ &\leq C_5 \left((1 - R)^\alpha (1 + R)^\beta + t^{2(\alpha-\beta)} (1 - R^2)^\beta \right. \\ &\quad \left. + t^{2(\beta-1)} (1 - R)^{\alpha-\beta+1} (1 + R) + t^{2(\alpha-1)} (1 - R^2) \right).\end{aligned}$$

Then clearly

$$I \leq C_6 \int_{-1}^1 |f(R)| \hat{\delta}_k(R) dR \quad (k = 1, 2).$$

The last inequality and the estimates for $\hat{\delta}_k(R)$, given above, yield

$$\begin{aligned}I \leq C_7 & \left(\|f\|_{1,\alpha,\beta} + t^{2\gamma_1} \|f\|_{1,\alpha-\gamma_1,\beta} + t^{2\gamma_3} \|f\|_{1,\alpha-\gamma_3,\beta-\gamma_3} \right. \\ & \left. + t^{2(\gamma_1+\gamma_3)} \|f\|_{1,\alpha-\gamma_1-\gamma_3,\beta-\gamma_3} \right),\end{aligned}$$

where the constant C_7 does not depend on f and t . Hence the lemma is true in the case $p = 1$.

Assume now that $1 < p < \infty$. Applying Hölder's inequality to the inside integral in (3.2) we get

$$\begin{aligned}I &= \int_{-1}^1 \left| \int_{-1}^1 |f(R)| (1 - z^2)^{-\frac{1}{2}-\frac{1}{p}+1-b} (1 - R^2) (1 - z^2)^{-(1-\frac{1}{p})+b} dz \right|^p \\ &\quad \times (1 - x)^{p(\alpha-1)} (1 + x)^{p(\beta-1)} dx \leq C_8 \int_{-1}^1 \int_{-1}^1 |f(R)|^p \varkappa dz dx,\end{aligned}$$

where

$$\varkappa = (1 - z^2)^{-1+p(\frac{1}{2}-b)} (1 - R^2)^p (1 - x^2)^{p(\beta-1)} (1 - x)^{p(\alpha-\beta)},$$

b is an arbitrary positive number, the constant C_8 does not depend on t and the function f .

Let $\beta < \frac{3}{2} - \frac{1}{2p}$. Put $b = \frac{3}{2} - \frac{1}{2p} - \beta$. Applying Lemma 3.9 we derive the estimate

$$\begin{aligned}\varkappa &\leq C_9 (1 - z^2)^{-1/2} ((1 - z^2) (1 - x^2))^{p(\beta-1)} (1 - R^2)^p (1 - R + t^2)^{p(\alpha-\beta)} \\ &= C_9 \varkappa_1(x, z, R).\end{aligned}$$

Let $\beta \geq \frac{3}{2} - \frac{1}{2p}$. Put $b = \varepsilon$, where ε is an arbitrary number belonging to the interval $0 < \varepsilon < \frac{1}{2}$. Again by Lemma 3.9 we see that

$$\begin{aligned}\varkappa &\leq C_{10} (1 - z^2)^{-1/2} ((1 - z^2) (1 - x^2))^{-\frac{1}{2}+p(\frac{1}{2}-\varepsilon)} (1 - R^2)^p \\ &\quad \times (1 - R^2 + t^2)^{p(\beta-\frac{3}{2}+\frac{1}{2p}+\varepsilon)} (1 - R + t^2)^{p(\alpha-\beta)} = C_{10} \varkappa_2(x, z, R).\end{aligned}$$

Using these estimates for \varkappa we get the inequality

$$I \leq C_{11} \int_{-1}^1 \int_{-1}^1 |f(R)|^p \varkappa_k(x, z, R) dz dx \quad (k = 1, 2).$$

and consequently (after the changes of variables (3.1)),

$$I \leq C_{11} \int_{-1}^1 \int_{-1}^1 |f(R)|^p \varkappa_k(R, V, R) dV dR \quad (k = 1, 2).$$

Set

$$\begin{aligned}\hat{\varkappa}_1(R) &:= (1 - R^2)^{p\beta} (1 - R + t^2)^{p(\alpha-\beta)} \\ &\leq C_{12} \left((1 - R)^{p\alpha} (1 + R)^{p\beta} + t^{2p(\alpha-\beta)} (1 - R^2)^{p\beta} \right),\end{aligned}$$

$$\begin{aligned}
\hat{\kappa}_2(R) &:= (1-R^2)^{-\frac{1}{2}+p(\frac{3}{2}-\varepsilon)} (1-R^2+t^2)^{p(\beta-\frac{3}{2}+\frac{1}{2p}+\varepsilon)} (1-R+t^2)^{p(\alpha-\beta)} \\
&\leq C_{13} \left((1-R)^{p\alpha}(1+R)^{p\beta} + t^{2p(\alpha-\beta)} (1-R^2)^{p\beta} \right. \\
&\quad \left. + t^{2p(\beta-\frac{3}{2}+\frac{1}{2p}+\varepsilon)} (1-R)^{p(\alpha-\beta+\frac{3}{2}-\frac{1}{2p}-\varepsilon)} (1+R)^{p(\frac{3}{2}-\frac{1}{2p}-\varepsilon)} \right. \\
&\quad \left. + t^{2p(\alpha-\frac{3}{2}+\frac{1}{2p}+\varepsilon)} (1-R^2)^{p(\frac{3}{2}-\frac{1}{2p}-\varepsilon)} \right).
\end{aligned}$$

Then clearly

$$I \leq C_{14} \int_{-1}^1 |f(R)|^p \hat{\kappa}_k(R) dR \quad (k=1, 2).$$

From the last inequality and the estimates of $\hat{\kappa}_k(R)$ we obtain

$$\begin{aligned}
I &\leq C_{15} \left(\|f\|_{p,\alpha,\beta}^p + t^{2p\gamma_1} \|f\|_{p,\alpha-\gamma_1,\beta}^p + t^{2p\gamma_3} \|f\|_{p,\alpha-\gamma_3,\beta-\gamma_3}^p \right. \\
&\quad \left. + t^{2p(\gamma_1+\gamma_3)} \|f\|_{p,\alpha-\gamma_1-\gamma_3,\beta-\gamma_3}^p \right),
\end{aligned}$$

where the constant C_{15} does not depend on f and t . This shows that the lemma is true in the case $1 < p < \infty$ as well.

Now let $p = \infty$. Consider the integral

$$\begin{aligned}
J &:= \int_{-1}^1 |f(R)| (1-z^2)^{-1/2} (1-R^2) (1-x)^{\alpha-1} (1+x)^{\beta-1} dz \\
&= \int_{-1}^1 |f(R)| \lambda dz,
\end{aligned}$$

where

$$\lambda = (1-z^2)^{-1+b} (1-R^2) (1-x^2)^{\beta-1} (1-x)^{\alpha-\beta} (1-z^2)^{\frac{1}{2}-b}$$

and b is an arbitrary positive number.

Let $\beta < \frac{3}{2}$. Put $b = \frac{3}{2} - \beta$. Applying the estimate from Lemma 3.9 we get

$$\begin{aligned}
\lambda &= (1-z^2)^{\frac{1}{2}-\beta} (1-R^2) ((1-z^2) (1-x^2))^{\beta-1} (1-x)^{\alpha-\beta} \\
&\leq C_{16} (1-z^2)^{\frac{1}{2}-\beta} (1-R^2)^\beta (1-R+t^2)^{\alpha-\beta} = C_{16} (1-z^2)^{\frac{1}{2}-\beta} \lambda_1(R).
\end{aligned}$$

Let $\beta \geq \frac{3}{2}$. Put $b = \varepsilon$, where ε is an arbitrary number from the interval $0 < \varepsilon < \frac{1}{2}$. Applying again Lemma 3.9 we see that

$$\begin{aligned}
\lambda &= (1-z^2)^{-1+\varepsilon} ((1-z^2) (1-x^2))^{\frac{1}{2}-\varepsilon} (1-R^2) (1-x^2)^{\beta-\frac{3}{2}+\varepsilon} (1-x)^{\alpha-\beta} \\
&\leq C_{17} (1-z^2)^{-1+\varepsilon} (1-R^2)^{\frac{3}{2}-\varepsilon} (1-R^2+t^2)^{\beta-\frac{3}{2}+\varepsilon} (1-R+t^2)^{\alpha-\beta} \\
&= C_{17} (1-z^2)^{-1+\varepsilon} \lambda_2(R).
\end{aligned}$$

Using these estimates for λ and taking into account the relations

$$\lambda_1(R) \leq C_{18} \left((1-R)^\alpha (1+R)^\beta + t^{2(\alpha-\beta)} (1-R^2)^\beta \right),$$

$$\begin{aligned}
\lambda_2(R) &\leq C_{19} \left((1-R)^\alpha (1+R)^\beta + t^{2(\alpha-\beta)} (1-R^2)^\beta \right. \\
&\quad \left. + t^{2(\beta-\frac{3}{2}+\varepsilon)} (1-R)^{\alpha-\beta+\frac{3}{2}-\varepsilon} (1+R)^{\frac{3}{2}-\varepsilon} + t^{2(\alpha-\frac{3}{2}+\varepsilon)} (1-R^2)^{\frac{3}{2}-\varepsilon} \right),
\end{aligned}$$

for $k = 1, 2$ we obtain

$$J \leq C_{20} \max_{-1 \leq R \leq 1} |f(R)| \lambda_k(R) \leq C_{21} (\|f\|_{\infty, \alpha, \beta} + t^{2\gamma_1} \|f\|_{\infty, \alpha - \gamma_1, \beta} + t^{2\gamma_3} \|f\|_{\infty, \alpha - \gamma_3, \beta - \gamma_3} + t^{2(\gamma_1 + \gamma_3)} \|f\|_{\infty, \alpha - \gamma_1 - \gamma_3, \beta - \gamma_3}),$$

where the constant C_{21} does not depend on f and t . This proves the lemma for $p = \infty$.

Thus, the lemma is proved for $\alpha \geq \beta$. The case $\alpha \leq \beta$ goes similarly. We omit the details. The proof is complete. \square

4.

Theorem 4.1. *Let p, α, β and γ be given numbers such that $1 \leq p \leq \infty$, $\gamma = \min\{\alpha, \beta\}$. Assume that*

$$\begin{aligned} \gamma &> 1 - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty, \\ \gamma &\geq 1 \quad \text{for } p = \infty. \end{aligned}$$

Let ε be an arbitrary number belonging to the interval $0 < \varepsilon < \frac{1}{2}$. Let

$$\gamma_1 = \begin{cases} \alpha - \beta, & \text{if } \alpha > \beta \\ 0, & \text{if } \alpha \leq \beta, \end{cases} \quad \gamma_2 = \begin{cases} 0, & \text{if } \alpha > \beta \\ \beta - \alpha, & \text{if } \alpha \leq \beta. \end{cases}$$

Set for $1 < p \leq \infty$

$$\gamma_3 = \begin{cases} \gamma - \frac{3}{2} + \frac{1}{2p} + \varepsilon, & \text{for } \gamma \geq \frac{3}{2} - \frac{1}{2p} \\ 0, & \text{for } \gamma < \frac{3}{2} - \frac{1}{2p}, \end{cases}$$

and

$$\gamma_3 = \begin{cases} \gamma - 1, & \text{for } \gamma \geq 1 \\ 0, & \text{for } \gamma < 1 \end{cases}$$

for $p = 1$. Then the following inequality holds true

$$\begin{aligned} \left\| \hat{T}_t(f, x) \right\|_{p, \alpha, \beta} &\leq C \left(\|f\|_{p, \alpha, \beta} + t^{2(\gamma_1 + \gamma_2)} \|f\|_{p, \alpha - \gamma_1, \beta - \gamma_2} \right. \\ &\quad \left. + t^{2\gamma_3} \|f\|_{p, \alpha - \gamma_3, \beta - \gamma_3} + t^{2(\gamma_1 + \gamma_2 + \gamma_3)} \|f\|_{p, \alpha - \gamma_1 - \gamma_3, \beta - \gamma_2 - \gamma_3} \right), \end{aligned}$$

where the constant C does not depend on f and t .

Proof. We have

$$\left\| \hat{T}_t(f, x) \right\|_{p, \alpha, \beta} \leq \frac{1}{\pi} \left\| \frac{1}{1 - x^2} \int_{-1}^1 A |f(R)| \frac{dz}{\sqrt{1 - z^2}} \right\|_{p, \alpha, \beta},$$

where $R = x \cos t - z \sqrt{1 - x^2} \sin t$,

$$A = 1 - R^2 - 2(1 - x^2) \sin^2 t + 4(1 - x^2)(1 - z^2)^2 \sin^2 t.$$

Using Lemma 3.9 we get

$$A \leq 1 - R^2 + 2(1 - R^2) + 4(1 - R^2)^2 \leq 7(1 - R^2).$$

Hence

$$\left\| \hat{T}_t(f, x) \right\|_{p, \alpha, \beta} \leq \frac{7}{\pi} \left\| \frac{1}{1 - x^2} \int_{-1}^1 (1 - R^2) |f(R)| \frac{dz}{\sqrt{1 - z^2}} \right\|_{p, \alpha, \beta}.$$

Now the theorem follows from Lemma 3.10. \square

Theorem 4.2. Let q , m and r be given natural numbers and let $f \in L_{1,2,2}$. The function

$$Q(x) = \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi (\Delta_{t_1, \dots, t_r}^r(f, x) - (-1)^r f(x)) \times \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \dots dt_r,$$

where

$$\gamma_m = \int_0^\pi A(t) \sin^3 t dt,$$

is an algebraic polynomial of degree not greater than $(q+2)(m-1)$.

Proof. To prove the theorem it is sufficient to show that for every $l = 1, \dots, r$ the function

$$Q^{(l)}(x) = \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi T_{\cos t_1, \dots, \cos t_l}^l(f, x) \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \dots dt_r$$

is an algebraic polynomial of degree not greater than $(q+2)(m-1)$.

It is obvious that the function $Q^{(l)}(x)$ can be written in the form

$$Q^{(l)}(x) = \frac{1}{(\gamma_m)^r} \left(Q_1^{(l)}(x) + \frac{3}{2} Q_2^{(l)}(x) \right),$$

where $Q_1^{(l)}(x)$ and $Q_2^{(l)}(x)$ are the functions from Lemmas 3.6 and 3.7, respectively. But then it follows from Lemmas 3.6 and 3.7 that $Q^{(l)}(x)$ is an algebraic polynomial of degree not greater than $(q+2)(m-1)$.

The theorem is proved. \square

Theorem 4.3. Let p , α , β , r and λ be given numbers such that $1 \leq p \leq \infty$, $\lambda > 0$, $r \in \mathbb{N}$. Assume that

$$\begin{aligned} \alpha &\leq 2, & \beta &\leq 2 & \text{for } p = 1, \\ \alpha &< 3 - \frac{1}{p}, & \beta &< 3 - \frac{1}{p} & \text{for } 1 < p \leq \infty. \end{aligned}$$

Let $f \in L_{p,\alpha,\beta}$ and

$$\hat{\omega}_r(f, \delta)_{p,\alpha,\beta} \leq M \delta^\lambda.$$

Then

$$E_n(f)_{p,\alpha,\beta} \leq CM n^{-\lambda},$$

where the constant C does not depend on f , M and n ($n \in \mathbb{N}$).

Proof. Under the conditions of the theorem, if $f \in L_{p,\alpha,\beta}$, then $f \in L_{1,2,2}$. Indeed, for $p = 1$ we have

$$\|f\|_{1,2,2} = \int_{-1}^1 |f(x)| (1-x)^\alpha (1+x)^\beta (1-x)^{2-\alpha} (1+x)^{2-\beta} dx \leq C_1 \|f\|_{1,\alpha,\beta}$$

provided $\alpha \leq 2$ and $\beta \leq 2$. For $1 < p < \infty$, by Hölder's inequality,

$$\begin{aligned} \|f\|_{1,2,2} &\leq \left\{ \int_{-1}^1 |f(x)|^p (1-x)^{p\alpha} (1+x)^{p\beta} dx \right\}^{1/p} \\ &\quad \times \left\{ \int_{-1}^1 (1-x)^{(2-\alpha)\frac{p}{p-1}} (1+x)^{(2-\beta)\frac{p}{p-1}} dx \right\}^{\frac{p-1}{p}} = C_2 \|f\|_{p,\alpha,\beta} \end{aligned}$$

for $\alpha < 3 - \frac{1}{p}$ and $\beta < 3 - \frac{1}{p}$. For $p = \infty$ we have

$$\|f\|_{1,2,2} \leq \|f\|_{\infty,\alpha,\beta} \int_{-1}^1 (1-x)^{2-\alpha} (1+x)^{2-\beta} dx = C_3 \|f\|_{\infty,\alpha,\beta}$$

provided $\alpha < 3$ and $\beta < 3$.

We choose a natural number q such that $2q > \lambda$, and for each $n \in \mathbb{N}$ we choose a number $m \in \mathbb{N}$ satisfying the condition

$$(4.1) \quad \frac{n-1}{q+2} < m \leq \frac{n-1}{q+2} + 1.$$

For these q and m the polynomial $Q(x)$ defined in Theorem 4.2 is from \mathcal{P}_n . Hence

$$\begin{aligned} E_n(f)_{p,\alpha,\beta} &\leq \|f(x) - (-1)^{r+1} Q(x)\|_{p,\alpha,\beta} \\ &= \left\| \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \Delta_{t_1, \dots, t_r}^r(f, x) \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \dots dt_r \right\|_{p,\alpha,\beta}. \end{aligned}$$

Applying the generalized inequality of Minkowski we obtain

$$\begin{aligned} E_n(f)_{p,\alpha,\beta} &\leq \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \|\Delta_{t_1, \dots, t_r}^r(f, x)\|_{p,\alpha,\beta} \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \dots dt_r \\ &\leq \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \sup_{|u_i| \leq \sum_{j=1}^r t_j} \|\Delta_{u_1, \dots, u_r}^r(f, x)\|_{p,\alpha,\beta} \\ &\quad \times \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \dots dt_r \\ &\leq \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \hat{\omega}_r\left(f, \sum_{j=1}^r t_j\right)_{p,\alpha,\beta} \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \dots dt_r. \end{aligned}$$

Hence, taking into account the assumptions of the theorem, we have

$$\begin{aligned} E_n(f)_{p,\alpha,\beta} &\leq \frac{M}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \left(\sum_{j=1}^r t_j\right)^\lambda \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \dots dt_r \\ &\leq C_4 M \sum_{j=1}^r \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi t_j^\lambda \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \dots dt_r. \end{aligned}$$

Applying the standard evaluation of Jackson's kernel and making use of inequality (4.1) we obtain

$$E_n(f)_{p,\alpha,\beta} \leq C_5 M m^{-\lambda} \leq C_6 M n^{-\lambda}.$$

Theorem 4.3 is proved. \square

Theorem 4.4. *Let p, α, β, r and λ be given numbers such that $1 \leq p \leq \infty, r \in \mathbb{N}$. Assume that*

$$\begin{aligned} \alpha &> 1 - \frac{1}{2p}, \quad \beta > 1 - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty, \\ \alpha &\geq 1, \quad \beta \geq 1 \quad \text{for } p = \infty; \end{aligned}$$

$$\lambda_0 = 2 \max \left\{ |\alpha - \beta|, \alpha - \frac{3}{2} + \frac{1}{2p}, \beta - \frac{3}{2} + \frac{1}{2p} \right\} < \lambda < 2r.$$

If $f \in L_{p,\alpha,\beta}$ and

$$E_n(f)_{p,\alpha,\beta} \leq \frac{M}{n^\lambda},$$

then

$$\hat{\omega}_r(f, \delta)_{p, \alpha, \beta} \leq CM\delta^\lambda,$$

where the constant C does not depend on f , M and δ .

Proof. Let $P_n(x)$ be the polynomial from \mathcal{P}_n for which

$$\|f - P_n\|_{p, \alpha, \beta} = E_n(f)_{p, \alpha, \beta} \quad (n = 1, 2, \dots).$$

We construct the polynomials $Q_k(x)$ by

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \dots)$$

and $Q_0(x) = P_1(x)$. Since for $k \geq 1$ we have

$$\begin{aligned} \|Q_k\|_{p, \alpha, \beta} &= \|P_{2^k} - P_{2^{k-1}}\|_{p, \alpha, \beta} \leq \|P_{2^k} - f\|_{p, \alpha, \beta} + \|f - P_{2^{k-1}}\|_{p, \alpha, \beta} \\ &= E_{2^k}(f)_{p, \alpha, \beta} + E_{2^{k-1}}(f)_{p, \alpha, \beta}, \end{aligned}$$

then it follows from the assumptions of the theorem that

$$\|Q_k\|_{p, \alpha, \beta} \leq C_1 M 2^{-k\lambda}.$$

It is obvious that without loss of generality we may assume that $t_s \neq 0$ ($s = 1, \dots, r$). Next we estimate the quantity

$$I = \|\Delta_{t_1, \dots, t_r}^r(f, x)\|_{p, \alpha, \beta}$$

for $0 < |t_s| < \delta$ ($s = 1, \dots, r$). For every natural number N , taking into account that the linearity of the operator $\hat{T}_{t_1}(f, x)$ implies the linearity of $\hat{T}_{t_1, \dots, t_r}^r(f, x)$, i.e. the linearity of the difference $\Delta_{t_1, \dots, t_r}^r(f, x)$, we have

$$I \leq \|\Delta_{t_1, \dots, t_r}^r(f - P_{2^N}, x)\|_{p, \alpha, \beta} + \|\Delta_{t_1, \dots, t_r}^r(P_{2^N}, x)\|_{p, \alpha, \beta}.$$

Since $P_{2^N}(x) = \sum_{k=0}^N Q_k(x)$, we get

$$I \leq \|\Delta_{t_1, \dots, t_r}^r(f - P_{2^N}, x)\|_{p, \alpha, \beta} + \sum_{k=0}^N \|\Delta_{t_1, \dots, t_r}^r(Q_k, x)\|_{p, \alpha, \beta} := A + \sum_{k=1}^N I_k.$$

Let N be chosen so that

$$(4.2) \quad \frac{\pi}{2^N} < \delta \leq \frac{\pi}{2^{N-1}}.$$

We shall show that

$$(4.3) \quad A \leq C_2 M \delta^\lambda$$

and

$$(4.4) \quad I_k \leq C_3 M \delta^{2r} 2^{k(2r-\lambda)}.$$

Consider first the quantity A . Assume that $r = 1$. An application of Theorem 4.1 to the function $\varphi(x) = f(x) - P_{2^N}(x)$ gives

$$\begin{aligned} \|\Delta_{t_1}(f - P_{2^N}, x)\|_{p, \alpha, \beta} &= \|\hat{T}_{t_1}(\varphi, x) - \varphi(x)\|_{p, \alpha, \beta} \\ &\leq \|\hat{T}_{t_1}(\varphi, x)\|_{p, \alpha, \beta} + \|\varphi(x)\|_{p, \alpha, \beta} \leq C_4 \left(\|\varphi\|_{p, \alpha, \beta} + \delta^{2(\gamma_1 + \gamma_2)} \|\varphi\|_{p, \alpha - \gamma_1, \beta - \gamma_2} \right. \\ &\quad \left. + \delta^{2\gamma_3} \|\varphi\|_{p, \alpha - \gamma_3, \beta - \gamma_3} + \delta^{2(\gamma_1 + \gamma_2 + \gamma_3)} \|\varphi\|_{p, \alpha - \gamma_1 - \gamma_3, \beta - \gamma_2 - \gamma_3} \right) \end{aligned}$$

for $|t_1| \leq \delta$, where the numbers γ_1 , γ_2 and γ_3 are chosen as in Theorem 4.1. Hence, by Lemma 3.1

$$\begin{aligned} \|\Delta_{t_1}(f - P_{2^N}, x)\|_{p, \alpha, \beta} &\leq C_5 M \left(2^{-N\lambda} + \delta^{2(\gamma_1 + \gamma_2)} 2^{-N(\lambda - 2\gamma_1 - 2\gamma_2)} \right. \\ &\quad \left. + \delta^{2\gamma_3} 2^{-N(\lambda - 2\gamma_3)} + \delta^{2(\gamma_1 + \gamma_2 + \gamma_3)} 2^{-N(\lambda - 2\gamma_1 - 2\gamma_2 - 2\gamma_3)} \right) \end{aligned}$$

for $\lambda > \lambda_0 + \varepsilon$, where the constant C_5 does not depend on f , M and δ . Here ε is either equal to 0 or is an arbitrary number belonging to the interval $0 < \varepsilon < \frac{1}{2}$. Therefore, this inequality holds for any $\lambda > \lambda_0$. Finally, applying inequality (4.2) we get

$$\|\Delta_{t_1}(f - P_{2^N}, x)\|_{p, \alpha, \beta} \leq C_6 M 2^{N\lambda} \leq C_7 M \delta^\lambda.$$

Thus inequality (4.3) is proved for $r = 1$.

Suppose that

$$\left\| \Delta_{t_1, \dots, t_{r-1}}^{r-1}(f - P_{2^N}, x) \right\|_{p, \alpha, \beta} \leq C_8 M \delta^\lambda.$$

Then inequality (4.2) yields

$$\begin{aligned} & \left\| \Delta_{t_1, \dots, t_{r-1}}^{r-1}(f - P_{2^N}, x) \right\|_{p, \alpha, \beta} \\ &= \left\| \Delta_{t_1, \dots, t_{r-1}}^{r-1}(f, x) - \Delta_{t_1, \dots, t_{r-1}}^{r-1}(P_{2^N}, x) \right\|_{p, \alpha, \beta} \leq C_9 \frac{M}{2^{N\lambda}}. \end{aligned}$$

Reasoning as above, i.e. applying first Theorem 4.1 to the function

$$\Delta_{t_1, \dots, t_{r-1}}^{r-1}(f - P_{2^N}, x),$$

taking into account that by Corollary 3.2 $\Delta_{t_1, \dots, t_{r-1}}^{r-1}(P_{2^N}, x)$ is an algebraic polynomial of degree not greater than $2^N - 1$, applying Lemma 3.1, and finally inequality (4.2), we obtain that

$$A = \left\| \Delta_{t_1, \dots, t_r}^r(f - P_{2^N}, x) \right\|_{p, \alpha, \beta} \leq C_{10} \delta^\lambda.$$

Inequality (4.3) is proved.

Now we prove inequality (4.4). Let

$$\psi_k(x) = \Delta_{t_1, \dots, t_r}^r(Q_k, x).$$

It can be shown that

$$\begin{aligned} \psi_k(x) = & \frac{1}{2\pi(1-x^2)} \int_0^{t_r} \int_{-u}^u \int_0^\pi \left(A(v)(R'_v)^2 \frac{d^2}{dR_v^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right. \\ & - (A(v)R_v - 2A'(v)R'_v) \frac{d}{dR_v} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \\ & \left. + A''(v) \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right) d\varphi dv du, \end{aligned}$$

where $R_v = x \cos v - \sqrt{1-x^2} \cos \varphi \sin v$,

$$A(v) = 1 - R_v^2 - 2 \sin^2 v \sin^2 \varphi + 4(1-x^2) \sin^2 v \sin^4 \varphi.$$

Applying the estimates from Lemma 3.9 and performing the change of variables $z = \cos \varphi$ we obtain

$$|\psi_k(x)| \leq \frac{C_{11}}{1-x^2} \int_0^{t_r} \int_{-u}^u \int_{-1}^1 B(R_v) \frac{dz}{\sqrt{1-z^2}} dv du,$$

where

$$\begin{aligned} B(R_v) = & (1 - R_v^2)^2 \left| \frac{d^2}{dR_v^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right| \\ & + (1 - R_v^2) \left| \frac{d}{dR_v} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right| + \left| \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right| \\ & = B_1(R_v) + B_2(R_v) + B_3(R_v). \end{aligned}$$

Therefore, using the generalized Minkowski inequality, we get

$$\begin{aligned}
 (4.5) \quad I_k = \|\psi_k(x)\|_{p,\alpha,\beta} &\leq C_{11} \int_0^{t_r} \int_{-u}^u \left\| \frac{1}{1-x^2} \int_{-1}^1 B(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} dv du \\
 &\leq C_{12} t_r^2 \sup_{|v| \leq t_r} \left\| \frac{1}{1-x^2} \int_{-1}^1 B(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \\
 &\leq C_{12} t_r^2 \sum_{\mu=1}^3 \sup_{|v| \leq t_r} \left\| \frac{1}{1-x^2} \int_{-1}^1 B_\mu(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta}.
 \end{aligned}$$

Next, applying first Lemma 3.10 to the function $B_1(R_v)$, then Lemma 3.2, and finally inequality (4.2), we get for $|v| \leq |t_r| \leq \delta$ and $k \leq N$

$$\begin{aligned}
 &\left\| \frac{1}{1-x^2} \int_{-1}^1 B_1(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \\
 &\leq C_{13} \left(\left\| (1-x^2) \frac{d^2}{dx^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta} \right. \\
 &\quad + |v|^{2(\gamma_1+\gamma_2)} \left\| (1-x^2) \frac{d^2}{dx^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-\gamma_1,\beta-\gamma_2} \\
 &\quad + |v|^{2\gamma_3} \left\| (1-x^2) \frac{d^2}{dx^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-\gamma_3,\beta-\gamma_3} \\
 &\quad \left. + |v|^{2(\gamma_1+\gamma_2+\gamma_3)} \left\| (1-x^2) \frac{d^2}{dx^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-\gamma_1-\gamma_3,\beta-\gamma_2-\gamma_3} \right) \\
 &\leq C_{14} (1 + |v|^{2(\gamma_1+\gamma_2)} 2^{2k(\gamma_1+\gamma_2)} + |v|^{2\gamma_3} 2^{2k\gamma_3} + |v|^{2(\gamma_1+\gamma_2+\gamma_3)} 2^{2k(\gamma_1+\gamma_2+\gamma_3)}) \\
 &\quad \times \left\| (1-x^2) \frac{d^2}{dx^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta} \\
 &\leq C_{15} \left\| \frac{d^2}{dx^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha+1,\beta+1}.
 \end{aligned}$$

Similarly, applying first Lemma 3.10, then Lemma 3.2, and finally inequality (4.2) we obtain

$$\left\| \frac{1}{1-x^2} \int_{-1}^1 B_2(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \leq C_{16} \left\| \frac{d}{dx} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta}$$

and

$$\left\| \frac{1}{1-x^2} \int_{-1}^1 B_3(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \leq C_{17} \left\| \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-1,\beta-1}.$$

Now, from inequality (4.5) and the fact that $|t_r| \leq \delta$ we derive the estimate

$$\begin{aligned}
 I_k &\leq C_{18} \delta^2 \left(\left\| \frac{d^2}{dx^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha+1,\beta+1} \right. \\
 &\quad \left. + \left\| \frac{d}{dx} \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta} + \left\| \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-1,\beta-1} \right).
 \end{aligned}$$

Applying twice Lemma 3.2 we obtain the recurrence relation

$$I_k = \left\| \Delta_{t_1, \dots, t_r}^r (Q_k, x) \right\|_{p,\alpha,\beta} \leq C_{18} \delta^2 2^{2k} \left\| \Delta_{t_1, \dots, t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta},$$

which yields

$$I_k \leq C_{19} \delta^{2r} 2^{2kr} \|Q_k\|_{p,\alpha,\beta} \leq C_{20} M \delta^{2r} 2^{k(2r-\lambda)}.$$

Inequality (4.4) is proved.

Now combining (4.3), (4.4) and (4.2) we finally get

$$I \leq C_{21}M \left(\delta^\lambda + \delta^{2r} \sum_{k=1}^N 2^{k(2r-\lambda)} \right) \leq C_{22}M \left(\delta^\lambda + \delta^{2r} 2^{N(2r-\lambda)} \right) \leq C_{23}M \delta^\lambda.$$

The proof of Theorem 4.4 is completed. \square

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